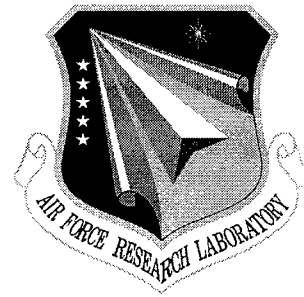AFRL-IF-RS-TR-1998-166
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STATISTICAL MODELING WITH IMPRECISE PROBABILITIES

SUNY Binghamton

George J. Klir

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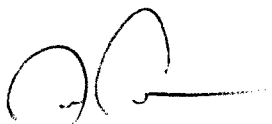
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2. INTRODUCTION

It has increasingly been recognized that practical applicability of classical probability theory to some problems of current interest, particularly those dependent on human judgment, is severely restricted. This restriction is a result of the additivity axiom of classical probability theory. Due to this axiom, elementary events are required to be pairwise disjoint and the probability of each is required to be expressed precisely by a real number in the unit interval [0,1]. The requirement of disjoint events makes mathematically good sense, but it becomes problematic whenever we leave the world of mathematics. As is well known, there is no method of scientific measurement that is free from error. As a consequence, observations in close neighborhoods of the sharp boundaries between events are unreliable and should be properly discounted. This results in a violation of the additivity axioms and, by implication, in a violation of the precision

requirement. Hence, we need to deal with imprecise probabilities. This need is even more pronounced when instead of measurements we are dependent on assessments based on subjective human judgment. There are other convincing arguments for imprecise probabilities, including, for example the following:

- Imprecision of probabilities is needed to reflect the amount of information on which they are based. The precision should increase with the amount of information available.

- Total ignorance can be properly modeled by vacuous probabilities, which are maximally imprecise (i.e., each covers the whole range [0,1]), but not by any precise probabilities.

- Imprecise probabilities are generally easier to assess and elicit than precise ones.

- We may be unable to assess probabilities precisely in practice, even if that is possible in principle, because we lack the time and computational ability.

- A precise probability model that is defined on some class of events determines only imprecise probabilities for events outside the class.

- When several sources of information (sensors, experts, and individuals in a group decision) are combined, the extent to which they are consistent can be reflected in the precision of the combined model.

The imprecision in expressing probabilities introduces a new dimension into statistical modeling. It can be utilized, for example, to reflect the amount of information upon which probabilities are estimated or the degree of conflict in information obtained from several different sources. These issues are crucial in statistical modeling with small observation sets as well as in problems of information fusion.

The first thorough study of imprecise probabilities was undertaken by Walley [1991]. His principal result is a demonstration that reasoning and decision making based on imprecise probabilities satisfies the principles of coherence and avoidance of sure loss, which are generally viewed as principles of rationality. Hence, the requirement of precision (or, equivalently, the additivity axiom) cannot be justified as inevitable for rationality, as previously believed. The soundness of this project is based on this important result.

In general, a probability model avoids sure loss if its use cannot lead to outcomes that are harmful to the user. It is coherent if it does not contain any internal inconsistencies resulting from the lack of transitivity of a relevant preference ordering. These concepts are carefully formalized by Walley [1991], and he shows that coherent models can be constructed from any set of probability assessments that avoid sure loss by a

mathematical procedure called a *natural extension*. Based on the perceived needs for dealing with imprecise probabilities, the objective of this project is to develop theoretical foundations and methodology for systems modeling with imprecise probabilities. It was realized that imprecise probabilities of different types must be distinguished since they require different methodological treatments. The imprecision of probabilities can be expressed, for example, by intervals of real numbers, convex sets of probability distributions, fuzzy numbers, or fuzzy intervals. In this project, these various types of imprecise probabilities are treated as special cases within fuzzy measure theory, a relatively new theory concerned with nonadditive measures, whose connection with imprecise probabilities is analogous to the connection of classical measure theory with precise probabilities.

The purpose of this report is to present a summary of results obtained by working on this project. It is organized as follows. An overview of basic issues addressed and type of results obtained in this project is presented in Sec. 3. Reviewing relevant background knowledge in Sec 4 follows this. The principal parts of the report are Sections. 5-8, in which the actual results obtained in the project are summarized at the conceptual level. These sections contain frequent references to 43 project-related publications, including two doctoral dissertations, where the results are presented in all technical details. The project-related publications are listed chronologically in Sec. 10, and numbers in square brackets makes references to them. Other references are listed in Sec. 9, and author's name and year of publication make references to them.

In all papers associated with this project, terminology and notation regarding fuzzy measures is adopted from [Wang and Klir 1992]. Terminology and notation regarding other areas is adopted from [Klir and Yuan, 1995].

Conclusions summarizing the significance of the obtained results are presented in Sec. 8, together with a discussion about prospective future directions of research pertaining to imprecise probabilities and related areas.

The work on the project was headed through the whole period by two faculty co-investigators, George J. Klir (Principal Investigator) and Zhenyuan Wang. Nine research assistants, Jiri Fridrich, David Harmanec, Yin Pan, Sylvia Rabeler, Jaron Rubenstein, John F. Swan-Stone, Jia Wang, Wei Wang, and Bo Yuan, worked on various aspects of the project and to various degrees.

3. PROJECT OVERVIEW

This project is based on the recognition that fuzzy measure theory relates to imprecise probabilities in a similar way as classical measure theory [Halmos 1950] relates to precise probabilities. Based on this recognition, it was decided to utilize fuzzy measure theory as a general framework within which imprecise probabilities can be investigated in rigorous and comprehensive way. Although the theory was fairly well developed when this project was initiated, research was still needed to make it fully operational for its intended use in this project.

Results obtained by working on this project are loosely classified into the following seven areas:

1. Complete methodology was developed for *Bayesian inference based on imprecise probabilities*, in which prior probabilities and likelihoods are either interval-valued or fuzzy [10,25,33,38]. This methodology includes a novel procedure for constructing fuzzy prior probabilities from opinions of selected experts. This procedures is based on pairwise comparisons of given alternatives according to each of several given criteria, as well as pairwise comparisons of the criteria [24,33].

2. Efficient algorithms were developed [17] for *computing theoretically well justified measures of uncertainty and uncertainty-based information* established by Harmanec and Klir [1994] in two theories of uncertainty. Both theories are distinct from classical probability theory and may be viewed as special theories of imprecise probabilities: Dempster-Shafer theory of evidence [Shafer 1976, Yager et al. 1994] and possibility theory [Dubois and Prade 1988, De Cooman 1997]. Based on these uncertainty measures, specific algorithms were also developed for information-preserving transformations [5] from Dempster-Shafer theory to either possibility theory or probability theory (approximations), and between possibility theory and probability theory (in both directions) [3, 18, 30]. The new results are integrated with relevant results obtained earlier by the principal investigator in three invited technical papers [19,29,31], a technical monograph [32], and an expository paper regarding the changing attitudes toward uncertainty and imprecision (including imprecise probabilities) in science [8]. Moreover, a well-justified measure of nonspecificity (one of several recognized types of uncertainty) was established for systems defined on the Euclidean space of any finite number of dimensions [6].

3. Numerous theoretical results were obtained in *fuzzy measure theory* in the process of utilizing the theory for investigating certain fundamental issues of imprecise probabilities, such as the issue of natural extensions of lower probabilities. Some of these results [1,9,26,37,42] directly contribute to the general theory of imprecise probabilities, as conceived by Walley [1991]. Other results in this category [14,15,16,27] are more fundamental to fuzzy measure theory itself, but were needed as prerequisites for pursuing research leading eventually to the results more directly connected with this project.

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4. The important issue of *constructing imprecise probabilities or associated fuzzy measures* under various scenarios was thoroughly investigated. Several distinct procedures emerged from this investigation [4,11,12,13,23,39,40]. The use of neural networks [23,40] and genetic algorithms [11,39] was found particularly fitting for this kind of problems.

5. *Various other results* emerged from the project. These results were obtained by broadening various special problems regarding imprecise probabilities. Each of these results opens a new direction for future research. They include: (*i*) Complete representations of Dempster-Shafer theory and possibility theory in terms of the usual semantics of propositional modal logic [3]; (*ii*) Basic ideas of mathematics of finite resolution, which is a general formalism for dealing with imprecision in all forms and, hence, is potentially relevant for dealing with imprecise probabilities [20,21]; (*iii*) Principles of constrained fuzzy arithmetic, which are needed for operating on fuzzy probabilities (e.g., in the Bayesian inference based on fuzzy probabilities) and are also essential in many other contexts [22,34,35]; (*iv*) A method for identifying key variables in data-driven systems modeling via fuzzy c-means clustering algorithm and an evolutionary program by which the distance function employed in the clustering is appropriately modified [7,23,36]; and (*v*) A thorough mathematical analysis of the well-known Cox's proof, by which it is shown that the proof does not demonstrate the inevitability of the rules of classical Bayesian inference, as often claimed [2,41].

6. *Two applications* of the Bayesian inference with interval-valued or fuzzy prior probabilities and likelihoods were developed [33]. The first application deals with the problem of military unit identification. While Bayesian inference is a proper tool for this problem, the requirement of classical Bayesian inference that all probabilities be precise is not realistic due to high observation uncertainties, which are in this case unavoidable. The second application deals with statistical decision making in face of imprecise prior probabilities and likelihoods.

7. *Computer programs* were developed to support some methodological tools resulting from the project. These are C++ programs implementing the following algorithms:

A. Four versions of Bayesian inference with imprecise probabilities [10,25,33,38]. The meaning of prior probabilities and likelihoods in each of these versions is shown in Table 1. It is clear that any combination that is not specifically handled by these versions, including precise prior probabilities, can be handled as a special case by the version closest to it.

Likelihoods	Prior probabilities	
	Intervals	Fuzzy
Precise	Version 1	Version 3
Intervals	Version 2	
Fuzzy		Version 4

Table 1: Meaning of the individual versions of programs developed for Bayesian inference with imprecise probabilities.

B. Computation of uncertainty measures in Dempster-Shafer theory [17]. Two versions were proposed: one suitable (computationally efficient) for bodies of evidence with small numbers of focal elements; the other suitable to those with large numbers of focal elements.

C. Computation of natural extensions with respect to given lower probabilities [37]. The concept of natural extension in the theory of imprecise probabilities is a generalization of the concept of expectation in classical probability theory.

D. Constructing λ-fuzzy measures [Sugeno 1974] from input-output data by a special genetic algorithm [11,39]. λ-measures are convenient for representing interval-valued probabilities.

Further information about these programs is covered in Appendix of this report.

4. BACKGROUND

4.1 Review of Fuzzy (Nonadditive) Measures

Given a universal set X and a non-empty family $\mathcal{C}$ of subsets of X (usually with an appropriate algebraic structure), a *fuzzy measure* (or *nonadditive measure*), μ, on $\langle X, \mathcal{C} \rangle$ is a function

$$\mu : \mathcal{C} \to [0, \infty]$$

that satisfies the following requirements:

    (μ1) $\mu (\varnothing)=0$ (vanishing at the empty set);

    (μ2) for all $A,B \in \mathcal{C}$, if $A \subseteq B$, then $\mu (A) \leq \mu (B)$ (monotonicity);

    (μ3) for any increasing sequence $A_1 \subseteq A_2 \ldots$ of sets in $\mathcal{C}$,

$$\text{if } \bigcup_{i=1}^{\infty} A_i \in \mathcal{C}, \text{ then } \lim_{i \to \infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i \right)$$

    (*continuity from below*);

(μ4) for any decreasing sequence $A_1 \supseteq A_2 \dots$ of sets in $\mathcal{C}$,

$$\text{if } \cap_{i=1}^{\infty} A_i \in \mathcal{C}, \text{ then } \lim_{i \to \infty} \mu(A_i) = \mu\left(\cap_{i=1}^{\infty} A_i\right)$$

(*continuity from above*).

Functions that satisfy requirements (μ1), (μ2), and either (μ3) or (μ4) are equally important in fuzzy measure theory. These functions are called semicontinuous from below or from above, respectively. When the universal set X is finite, requirements (μ3) and (μ4) are trivially satisfied and may thus be disregarded.

When fuzzy measures are utilized for characterizing uncertainty (e.g., lower or upper probabilities), their domain $\mathcal{C}$ is usually a σ-algebra of subsets of X (often the full power set, $\mathcal{P}(X)$, of X), their range is the unit interval [0,1], and it is required that $\mu(X)=1$.

For any pair A, B $\in$ $\mathcal{C}$ such that A $\cap$ B = $\varnothing$, a fuzzy measure μ is capable of capturing any of the following situations with respect to the measured property:

(a) $\mu(A \cup B) > \mu(A) + \mu(B)$ (*superadditivity*), which expresses a cooperative action or synergy between A and B;

(b) $\mu(A \cup B) = \mu(A) + \mu(B)$ (*additivity*), which expresses the fact that A and B are noninteractive;

(c) $\mu(A \cup B) < \mu(A) + \mu(B)$ (*subadditivity*), which expresses some sort of incompatibility between A and B.

Observe that classical probability theory, which is based on classical measure theory [Halmos 1950], is capable of capturing only situation (b). This demonstrates that fuzzy measure theory provides us with a considerably broader framework than probability theory for formalizing uncertainty. As a consequence, it allows us to capture types of uncertainty that are beyond the scope of probability theory.

A special class of fuzzy measures that can be utilized for expressing imprecise probabilities are measures that satisfy the following special axiom: for all A, B $\in$ $\mathcal{C}$, if A $\cap$ B = $\varnothing$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda \mu(A)\mu(B)$$

where $\lambda \in (-1, \infty)$ is a parameter by which different measures in this class are distinguished. These measures, which are easy to handle, are usually called λ-*fuzzy measures* or *Sugeno measures* [Sugeno 1974]. These measures are superadditive when $\lambda > 0$, additive when $\lambda = 0$, and subadditive when $\lambda < 0$.

4.2 Review of Dempster-Shafer Theory (DST)

DST is based on two dual nonadditive measures: belief measures and plausibility measures. Given a universal set X (usually referred to as the *frame of discernment* in DST), assumed here to be finite, a *belief measure* is a function

$$Bel : \mathcal{P}(X) \to [0,1]$$

such that $Bel(\varnothing) = 0$, $Bel(X) = 1$, and

$$Bel(A_1 \cup A_2 \cup \cdots \cup A_n) \geq \sum_j Bel(A_j) - \sum_{j<k} Bel(A_j \cap A_k)$$
$$+ \cdots + (-1)^{n+1} Bel(A_1 \cap A_2 \cap \cdots \cap A_n) \tag{1}$$

for all possible families of subsets of X. Due to the inequality (1), belief measures are *superadditive*. When X is infinite, function Bel is also required to be *continuous from above*. A *plausibility measure* is a function

$$Pl : \mathcal{P}(X) \to [0,1]$$

such that $Pl(\varnothing) = 0$, $Pl(X) = 1$, and

$$Pl(A_1 \cap A_2 \cap \cdots \cap A_n) \leq \sum_j Pl(A_j) - \sum_{j<k} Pl(A_j \cup A_k) \tag{2}$$
$$+ \cdots + (-1)^{n+1} Pl(A_1 \cup A_2 \cup \cdots \cup A_n)$$

for all possible families of subsets of X. Due to the inequality (2), plausibility measures are *subadditive*. When X is infinite, function Pl is also required to be *continuous from below*.

It is well known that either of the two measures is uniquely determined from the other one by the equation

$$Pl(A) = 1 - Bel(\overline{A}) \tag{3}$$

for all $A \in \mathcal{P}(X)$. It is also well known that

$$Pl(A) \geq Bel(A) \tag{4}$$

for each $A \in \mathcal{P}(X)$. In the special case of equality in (4) for all $A \in \mathcal{P}(X)$, we obtain a classical (additive) probability measure.

DST, which has become an important tool for dealing with uncertainty [Yager, et al. 1994], is best covered in a book by Shafer [1976]. Its position in fuzzy measure theory is described by Wang and Klir [38].

Belief and plausibility measures can conveniently be characterized by a function

$$m: \mathcal{P}(X) \rightarrow [0,1],$$

which is required to satisfy two conditions:

(i) $m(\emptyset) = 0$;

(ii) $\sum_{A \in \mathcal{P}(X)} m(A) = 1.$

This function is called a *basic probability assignment*. For each set $A \in \mathcal{P}(X)$, the value $m(A)$ expresses the proportion to which all available and relevant evidence supports the claim that a particular element of X, whose characterization in terms of relevant attributes is deficient, belongs to the set A. This value, $m(A)$, pertains solely to one set, set A; it does not imply any additional claims regarding subsets of A. If there is some additional evidence supporting the claim that the element belongs to a subset of A, say $B \subset A$, it must be expressed by another value $m(B)$.

Given a basic probability assignment m, the corresponding belief measure and plausibility measure are determined for all sets $A \in \mathcal{P}(X)$ by the formulas

$$Bel(A) = \sum_{B \mid B \subseteq A} m(B), \qquad (5)$$

$$Pl(A) = \sum_{B \mid A \cap B \neq \emptyset} m(B). \qquad (6)$$

Inverse procedure is also possible. Given, for example, a belief measure Bel, the corresponding basic probability assignment m is determined for all $A \in \mathcal{P}(X)$ by the formula

$$m(A) = \sum_{B \mid B \subseteq A} (-1)^{|A-B|} Bel(B), \qquad (7)$$

as proven by Shafer [1976]. Hence, each of the three functions, m, Bel and Pl, is sufficient to determine the other two.

Given a basic probability assignment, every set $A \in \mathcal{P}(X)$ for which $m(A) \neq 0$ is called a *focal element*. The pair $(\mathcal{F}, m)$, where $\mathcal{F}$ denotes the set of all focal elements induced by m is called a *body of evidence*.

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Total ignorance is expressed in evidence theory by $m(X) = 1$ and $m(A) = 0$ for all $A \neq X$. Full certainty is expressed by $m(\{x\}) = 1$ for one particular element of X and $m(A) = 0$ for all $A \neq \{x\}$.

To investigate ways of measuring the amount of uncertainty represented by each body of evidence (Sec. 6.2), it is essential to understand properties of bodies of evidence whose focal elements are subsets of the Cartesian product of two sets. That is, we need to examine basic probability assignments of the form

$$m : \mathcal{P}(X \times Y) \rightarrow [0,1],$$

where X and Y denote universal sets pertaining to two distinct domains of inquiry (e.g., two investigated variables), which may be connected in some fashion. Let m of this form be called a *joint basic probability assignment.*

In this case, each focal element induced by m is a binary relation R on $X \times Y$. When R is projected on set X and on set Y, we obtain, respectively, the sets

$$R_X = \{x \in X \mid (x,y) \in R \text{ for some } y \in Y\}$$
and
$$R_Y = \{y \in Y \mid (x,y) \in R \text{ for some } x \in X\}.$$

These sets are instrumental in calculating *marginal basic probability assignments* m_X and m_Y from the given joint assignment m:

$$m_X(A) = \sum_{R \mid A = R_X} m(R) \text{ for all } A \in \mathcal{P}(X),$$

$$m_Y(B) = \sum_{R \mid B = R_Y} m(R) \text{ for all } B \in \mathcal{P}(Y).$$

Let bodies of evidence associated with m_X and m_Y, $(\mathcal{F}_X, m_X)$ and $(\mathcal{F}_Y, m_Y)$, be called marginal bodies of evidence. These bodies are said to be *noninteractive* if and only if for all $A \in \mathcal{F}_X$ and all $B \in \mathcal{F}_Y$

$$m(A \times B) = m_X(A) \cdot m_Y(B) \tag{8}$$
and
$$m(R) = 0 \text{ for all } R \neq A \times B.$$

It is well known [Shafer 1976] that the belief measure and its dual plausibility measure collapse into a single measure whenever all focal elements are singletons. This measure is additive and, hence, it is a *probability measure.*

Any probability measure, Pro, on a finite set X can be uniquely determined by a *probability distribution function*

$$p : X \rightarrow [0,1]$$

via the formula

$$Pro(A) = \sum_{x \in A} p(x).$$

From the standpoint of evidence theory, clearly

$$p(x) = m(\{x\})$$

for all $x \in X$.

Probability measures are special belief measures (as well as plausibility measures) and, hence, are subject to the same rules as the latter. The various special features of probability theory are well known and are not covered here.

It is well known [Wang and Klir 1992] that λ-fuzzy measures introduced in Sec.4.1 are closely associated with belief and plausibility measures of DST. Each λ-fuzzy measure is also a belief measure when $\lambda > 0$, a plausibility measure when $\lambda < 0$, and a probability measure when $\lambda = 0$. Although λ-fuzzy measures do not capture all belief and plausibility measures, they are considerably easier to represent and manipulate. Hence, they are often used instead of DST to reduce computational complexity.

4.3 Review of Possibility Theory

Possibility theory is based on two dual semicontinuous fuzzy measures, called possibility measures and necessity measures. A *possibility measure*, Pos, is characterized by the property

$$Pos(\bigcup_{i \in I} A_i) = \sup_{i \in I} Pos(A_i) \tag{9}$$

for any family of subsets of X defined by an arbitrary index set I, where sup denotes the supremum (least upper bound). All possibility measures are semicontinuous from below. A *necessity measure*, Nec, is characterized by the property

$$Nec(\bigcap_{i \in I} A_i) = \inf_{i \in I} Nec(A_i) \tag{10}$$

for any family of subsets of X, where inf denotes the infimum (greatest lower bound). All necessity measures are semicontinuous from above.

Given a possibility measure Pos, a function r on X defined by the equation

$$r(x) = Pos(\{x\}) \tag{11}$$

for all $x \in X$ is called a *possibility distribution function*. This function uniquely characterizes the possibility measure via the formula

$$Pos(A) = \sup_{x \in A} r(x) \tag{12}$$

for all $A \in \mathcal{P}(X)$. Moreover, given a possibility measure Pos, there exists a necessity measure, Nec, that is dual with respect to Pos in the sense that

$$Nec(A) = 1 - Pos(\overline{A}) \tag{13}$$

for all $A \in \mathcal{P}(X)$. Several other important properties of possibility and necessity measures can readily be derived [Dubois and Prade 1988]:

$$Pos (A \cap B) \le \min[Pos(A), Pos(B)], \tag{14}$$
$$Nec(A \cup B) \ge \max[Nec(A), Nec(B)], \tag{15}$$
$$Pos(A) < 1 \Rightarrow Nec(A) = 0 \tag{16}$$
$$Nec(A) > 0 \Rightarrow Pos(A) = 1 \tag{17}$$

for all $A, B \in \mathcal{P}(X)$.

For the sake of simplicity, let us now assume that X is a finite set. Let $X = \{x_1, x_2, \cdots, x_n\}$ and let r_i denote $r(x_i)$ for all $i = 1, 2, \ldots, n$. Assume further (without any loss of generality) that $r_i \ge r_{i+1}$ for all $i = 1, 2, \ldots, n - 1$. Then the basic probability assignment function m associated with r is given by the formula

$$m(A) = \begin{cases} r_i - r_{i+1} & \text{for } A = E_i, \, i = 1, 2, \cdots, n \\ 0 & \text{for } A \ne E_i \text{ for all } i = 1, 2, \cdots, n \end{cases} \tag{18}$$

for all $A \in \mathcal{P}(X)$, where $E_i = \{x_1, x_2, \ldots, x_i\}$ for all $i = 1, 2, \ldots, n$ and $r_{n+1} = 0$ by convention. The inverse formula is

$$r_i = \sum_{k=i}^{n} m(E_k) \tag{19}$$

for all $i = 1, 2, \ldots, n$ [Klir and Yuan 1995].

Possibility theory may be viewed as a special branch of DST, in which focal elements are always nested [Shafer 1976]. This makes possibility theory naturally suited for representing evidence in terms of fuzzy sets, since α-cuts of fuzzy sets are also nested (Sec. 4.4). To explain this connection, let $\mathcal{X}$ denote a variable that takes values in a universal set X, and let information about the actual value of the variable be expressed by a fuzzy proposition "$\mathcal{X}$ is F," where F is a fuzzy set on X. To express this information in

measure-theoretic terms, it is natural to interpret the membership degree F(x) for each x ∈ X as the degree of possibility that $\mathcal{X}$ = x. This interpretation induces a possibility distribution function, r_F, on X that is defined by the equation

$$r_F(x) = F(x)+1-h_F \qquad (20)$$

for all x ∈ X, where

$$h_F = \sup_{x \in X} F(x)$$

Eq. (20) applies to both normal fuzzy sets (h_F=1) and subnormal fuzzy sets (h_F<1) [Klir 1998].

4.4 Review of Fuzzy Probabilities

To define *fuzzy probabilities*, we need to explain first the concept of a fuzzy set. In general, boundaries of fuzzy sets are not required to be precise. Each fuzzy set is fully characterized by a function that assigns to each element of a designated universal set its *degree of membership* in the fuzzy set, usually in the unit interval [0,1]. This function is called a *membership function*. A fuzzy set A is thus usually expressed in terms of a membership function of the form

$$A: X \rightarrow [0,1],$$

where X is a crisp (non fuzzy) universal set under consideration. For each x∈X, the value A(x) designates the degree of membership of x in A or, alternatively, the degree of compatibility of X with the concept represented by A.

The largest membership degree in a given fuzzy set A is called its *height* and is denoted by h_A. More precisely,

$$h_A = \sup_{x \in X} A(x), \qquad (21)$$

where sup denotes the supremum. When h_A=1, A is called a *normal fuzzy set*; otherwise, it is called *subnormal*. When A is defined on a finite universal set X, the number

$$|A| = \sum_{x \in X} A(x) \qquad (22)$$

is called a *scalar cardinality* of A or, by some authors, a *sigma count* of A.

For every α∈[0,1], a given fuzzy set A yields the crisp set

$$^{\alpha}A = \left\{ x \in X \mid A(x) \geq \alpha \right\}, \tag{23}$$

which is called an α-cut. Since $\alpha_1 < \alpha_2$ implies $^{\alpha_1}A \supseteq {}^{\alpha_2}A$, the family of all distinct α-cuts of any fuzzy set forms a nested sequence of crisp sets. This sequence uniquely represents the given fuzzy set via the formula

$$A(x) = \sup_{\alpha \in [0,1]} \alpha \cdot {}^{\alpha}A(x) \tag{24}$$

for all $x \in X$ [Klir and Yuan 1995]. This connection between fuzzy sets and crisp sets makes it possible to generalize properties of crisp sets into their fuzzy counterparts. A property of crisp sets is generalized into fuzzy sets when it is preserved (in the classical, crisp sense) in all α-cuts of relevant fuzzy sets. Properties of classical sets that are generalized into fuzzy sets in this way are called *cutworthy*. It is important to realize that only some properties of fuzzy sets are cutworthy.

Fuzzy sets that are defined on the set of real numbers, $\mathbb{R}$, are called fuzzy quantities. Some fuzzy quantities are particularly useful since they can be employed as meaningful approximators of closed intervals of real numbers or individual real numbers. They are called fuzzy intervals and fuzzy numbers, respectively. A *fuzzy interval* is any normal fuzzy quantity with bounded support whose α-cuts for all $\alpha \in (0,1]$ are closed intervals of real numbers [Klir and Yuan, 1995}. A fuzzy number A is a special fuzzy interval for which $A(x)=1$ for exactly one $x \in \mathbb{R}$.

Each fuzzy interval may conveniently be expressed for all $x \in \mathbb{R}$ in the canonical form

$$A(x) = \begin{cases} f_A(x) & \text{when } x \in [a,b) \\ 1 & \text{when } x \in [b,c] \\ g_A(x) & \text{when } x \in (c,d] \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

where $a, b, c, d \in \mathbb{R}$ are such that $a \leq b \leq c \leq d$, f_A is a real-valued function that is increasing and right-continuous, and g_A is a real-valued function that is decreasing and left-continuous.

For any fuzzy interval A expressed in the canonical form of Eq. (25), the α-cuts of A are expressed for all $\alpha \in (0,1]$ by the formula

$$^{\alpha}A = \begin{cases} \left[f_A^{-1}(\alpha), g_A^{-1}(\alpha) \right] & \text{when } \alpha \in (0,1) \\ [b,c] & \text{when } \alpha = 1, \end{cases} \tag{26}$$

where f_A^{-1} and g_A^{-1} are the inverse functions of f_A and g_A, respectively.

When functions f_A and g_A in (25) are linear functions, A is called a *trapezoidal fuzzy interval*. This type is often employed in applications since it is easy to represent and manipulate. Each trapezoidal fuzzy interval is fully characterized by the quadruple $\langle$a, b, c, d$\rangle$ of real numbers in (25) via the special canonical form

$$A(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{when } x \in [a,b) \\ 1 & \text{when } x \in [b,c] \\ \dfrac{d-x}{d-c} & \text{when } x \in (c,d] \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

It is convenient to use A= $\langle$a, b, c, d$\rangle$ as a shorthand symbol representing this special form. When b=c, we obtain a *triangular fuzzy number*.

It follows immediately form Eq. (26) and Eq. (27) that α-cuts of trapezoidal fuzzy intervals are expressed for all $\alpha \in (0,1]$ by the formula

$$^{\alpha}A = [a + (b-a)\alpha, d - (d-c)\alpha]. \tag{28}$$

Fuzzy intervals and fuzzy numbers are important for this project since each fuzzy probability is a fuzzy interval or a fuzzy number defined on the unit interval [0,1]. In general, we can apply fuzzy arithmetic on fuzzy intervals (or numbers) and, hence, on fuzzy probabilities [Kaufman and Gupta 1985, Klir and Yuan 1995]. However, since probabilities must add to one and due to some other reasons explained in Sec. 7.2, standard fuzzy arithmetic is not applicable and must be replaced with *constrained fuzzy arithmetic* [22,34,35].

5. BAYESIAN INFERENCE WITH IMPRECISE PROBABILITIES

5.1 Interval-Valued Probabilities

Interval-valued probabilities are perhaps the most common imprecise probabilities. They are easy to represent and manipulate, and have a meaningful and useful interpretation in many applications. In addition, they are essential for formulating and dealing with fuzzy probabilities, as explained in Sec. 5.2.

Results regarding Bayesian inference with interval-valued probabilities, which are only summarized in this section, are presented in detail in Refs. [33, 38].

Let S be a finite *sample space* with a *probability measure* μ, and let $\mathcal{X}$ be a discrete *random variable* with values in the set $X=\{x_i \mid i \in \mathbb{N}_n\}$, where $\mathbb{N}_n=\{1,2,\ldots,n\}$. Subsets of S are called events. For any $A \subseteq S$, $\mu(A)$ denotes the probability of event A. For each $x_i \in X$, there is a unique event

$$A_i = \{s \in S \mid \mathcal{X}(s) = x_i\},$$

in terms of which we define the probability of $\mathcal{X}=x_i$, denoted by $p(x_i)$, as

$$p(x_i) = \mu(A_i).$$

Then, the n-tuple

$$\mathbf{p} = \langle p(x_i) \mid i \in \mathbb{N}_n \rangle$$

is called a *probability distribution* on X. For the sake of simplicity, let $a_i = p(x_i)$ for all $i \in \mathbb{N}_n$.

Assume now that each of the probabilities $p(x_i)=a_i$ in $\mathbf{p}$ is not known exactly, but we know its lower estimate, $\underline{a}_i$, and upper estimate, $\overline{a}_i$. That is, we only know that

$$a_i \in [\underline{a}_i, \overline{a}_i].$$

Then, the probability distribution $\mathbf{p}$ becomes a *tuple of probability intervals* (TPI)

$$I = \langle [\underline{a}_i, \overline{a}_i] \mid i \in \mathbb{N}_n \rangle$$

Each TPI represents a set of probability distributions, P_I, defined as

$$P_I = \left\{ \langle a_i \mid i \in \mathbb{N}_n \rangle \mid a_i \in [\underline{a}_i, \overline{a}_i] \text{ for all } i \in \mathbb{N}_n \text{ and } \sum_{i \in \mathbb{N}_n} a_i = 1 \right\}.$$

We say that a given TPI I is *reasonable* if its associated set of probability distributions P_I is not empty. Whenever P_I is empty, this indicates that the estimates employed in I are conflicting with one another and, hence, they are totally unusable. In such a case, the assessments must be properly revised.

An important subset of reasonable TPI's are called feasible. A reasonable TPI I is feasible if for each $i \in \mathbb{N}_n$ and every $z \in [\underline{a}_i, \overline{a}_i]$ there exists at least one probability distribution in P_I such as $a_i = z$.

For any reasonable TPI,

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$$I = \left\langle [\underline{a}_i, \overline{a}_i] \mid i \in \mathbb{N}_n \right\rangle,$$

there exists a unique feasible TPI

$$I' = \left\langle [\underline{a}'_i, \overline{a}'_i] \mid i \in \mathbb{N}_n \right\rangle$$

such that $P_{I'} = P_I$. This unique TPI is obtained for all $i \in \mathbb{N}_n$ by the formulas

$$\underline{a}'_i = \max\left(\underline{a}_i, \ 1 - \sum_{j \neq i} \overline{a}_j \right) \tag{29}$$

$$\overline{a}'_i = \max\left(\overline{a}_i, \ 1 - \sum_{j \neq i} \underline{a}_j \right). \tag{30}$$

Clearly, $[\underline{a}'_i, \overline{a}'_i] \subseteq [\underline{a}_i, \overline{a}_i]$ for all $i \in \mathbb{N}_n$. That is, among all reasonable TPI's that are associated with the same set of probability distributions, the unique feasible TPI is defined in terms of the smallest intervals.

In the developed Bayesian inference based on interval-valued probabilities [38], prior probabilities are always interval-valued while likelihoods are either precise or interval-valued.

Applying the formula of Bayesian inference to interval-valued probabilities means that interval arithmetic must be used. However, the standard interval arithmetic [Moore 1966] is not directly applicable due to the requirement of probability theory that probabilities in each probability distribution add to 1. Hence constrained interval arithmetic must be employed (Sec. 7.2).

Given a prior TPI that is feasible and likelihoods (exact or interval-valued), it was proven by Pan and Klir [38] that the corresponding posterior TPI can always be determined by two linear programming problems. These problems are employed, respectively, for calculating lower and upper bounds of the posterior TPI. It was also proven that the posterior TPI obtained by solving the linear programming problems are always feasible.

Formulation of the linear programming problems, all supporting mathematical proofs, and various related algorithmic considerations are covered in Refs. [33, 38].

5.2 Fuzzy Probabilities

Bayesian inference in terms of fuzzy probabilities, as developed in this project [10, 25, 33], is a generalization of results obtained for Bayesian inference with interval-valued probabilities. To connect the two versions, the α-cut representation of fuzzy probabilities was employed (Sec. 4.4).

Bayesian inference with fuzzy probabilities deals with *tuples of fuzzy probabilities* (TFP), which for each $\alpha \in (0,1]$ become TPI's

$$^{\alpha}I = \left\langle [\underline{a}_i(\alpha), \overline{a}_i(\alpha)] \mid i \in \mathbb{N}_n \right\rangle.$$

That is, each lower bound $\underline{a}_i$ and upper bound $\overline{a}_i$ in a TFP is a function of α. For each $i \in \mathbb{N}_n$, the two functions, $\underline{a}_i(\alpha)$ and $\overline{a}_i(\alpha)$, define a fuzzy probability by Eq.(26) via the canonical form (25) introduced in Sec. 4.4.

The computation of fuzzy posterior probabilities is based on similar linear programming problems as the computation of interval-valued posterior probabilities. However, the former linear programming problems involve, in addition, the parameter α that keeps track of the meaning of each TPI in terms of the associated TFP.

For the sake of computational efficiency, fuzzy prior probabilities, as well as likelihoods are assumed to be trapezoidal fuzzy intervals or triangular fuzzy numbers. Due to the nature of processing of the fuzzy probabilities, these shapes are usually not fully preserved in the resulting fuzzy posterior probabilities. To employ the posterior probabilities obtained in a particular step of Bayesian inference as prior probabilities in the next step, it is desirable (for the sake of computational efficiency) to approximate them by trapezoidal fuzzy intervals or triangular fuzzy numbers. A simple approximation, which has commonly been used for this purpose, is obtained by keeping the values a, b, c, d in (25) unchanged and replacing functions f_A and g_A in (25) with their linear counterparts

$$f_A(x) = \frac{x-a}{b-a} \text{ and } g_A(x) = \frac{d-x}{d-c}.$$

The accumulated error resulting from this approximation for repeated use of the multiplication operation was examined by Giacheti and Young [1998]. They showed that the error can be sufficiently large to produce misleading results. They proposed an alternative approximation and showed that it leads to acceptable results in most practical cases. Moreover, they derived an error expression for the new method, which can be used for checking whether or not the obtained results are acceptable.

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Another approximation was proposed by Pan [33]. It is based on keeping only values b and c in (25) unchanged, and calculating new values for a and d in (25) in terms of a weighted least-square-error; the weights are monotone increasing with values of α according to some rule. Although this approximation method seems promising, no error analysis has been made for it yet.

5.3 Assessment of Prior Probabilities

An important issue in Bayesian inference is the *determination of prior probabilities*, particularly when this involves human judgment. This issue was addressed by Saaty [1980] in terms of his well-known method of hierarchical analysis. In this method, selected experts are asked to make pairwise ratio comparisons of given alternatives for each of several relevant criteria, as well as pairwise ratio comparisons of the criteria themselves. These comparisons are then appropriately processed to obtain overall pairwise ratio comparisons of the alternatives and a value of an index by which the degree of consistency of the given experts' judgments is expressed.

In this project, a counterpart of Saaty's method was developed, which is based on *differences rather than ratios* [24, 33]. An advantage of this method is its significantly lower computational complexity since all operations of multiplication and division in Saaty's method are replaced, respectively, with the operations of addition and subtraction in Pan's method. The method was also generalized to allow the experts to express their comparisons in terms of fuzzy numbers. This produces either fuzzy prior probabilities or, when defuzzified, precise prior probabilities.

6. DEMPSTER-SHAFER THEORY (DST)

6.1 Construction of Belief Measures

Belief measures and *plausibility measures*, the two dual nonadditive measures of DST, are commonly interpreted as lower and upper probabilities, respectively. Although DST is now mathematically well developed [Shafer 1976, Guan and Bell 1991-92] and has a great potential applicability in many areas [Yager et al. 1994], its practical utility has been hampered by the lack of efficient methods for constructing belief measures (or the dual plausibility measures) in the context of specific applications. To alleviate this weakness in the applicability of DST, relevant research was conducted as a part of this project.

The problem of *constructing belief measures* was recognized as a special case of the broader problem of constructing arbitrary fuzzy measures. This makes sense since imprecise probabilities can be represented within fuzzy measure theory in more than one way. Six distinct strategies for constructing fuzzy measures were investigated:

- Constructing a new fuzzy measure from a given fuzzy measure by the Sugeno integral or the Choquet integral and a suitable class of parameterized functions [4,14,15];

- Constructing new fuzzy measures from given fuzzy measures by suitable transformations [12,13];

- Constructing fuzzy measures from data by special neural networks or genetic algorithms [11,23,39,40,42];

- Constructing various types of fuzzy measures via their representations in terms of the usual semantics of propositional modal logic [3];

- Constructing fuzzy measures on given classes of sets by extensions from fuzzy measures defined on smaller classes [1,26,37];

- Constructing fuzzy measures from application constraints by using the principle of maximum uncertainty [5].

Technical summary of these broad investigations, only some of which were directly connected with this project, is presented in an expository paper by Klir, Wang, and Harmanec [1997].

Among these broad investigations of methods for constructing fuzzy measures, two methods resulting from these investigations are explicitly oriented to constructing specific imprecise probabilities. One of them is a powerful method for constructing belief measures from input-output data by learning via a special neural network [40]. The other one is a method for constructing Sugeno measures (Sec. 4.1) via a special genetic algorithm [39]. As is well known, Sugeno measures capture a significant subset of the dual pairs of belief and plausibility measures of DST, but they are considerably easier to represent and manipulate.

The first method is based on a representation of the Choquet integral by a special nonlinear neural network with five layers [40]. The Choquet integral, $\int f d\mu$, of a nonnegative real-valued function f on X with respect to a fuzzy measure μ on the power set of X is defined by the formula

$$\int f d\mu = \int_0^1 \mu(^\alpha F) d\alpha,$$

where $^\alpha F = \{x \in X | f(x) \geq \alpha\}$ for any $\alpha \in [0, 1]$. When μ is a classical measure (i.e., additive), the Choquet integral coincides with the Lebesque integral. When μ is a belief measure (or plausibility measure), the Choquet integral is a coherent lower prevision (or

upper prevision, respectively) in the sense of the theory of imprecise probabilities [1, 26]. Given a set of input-output pairs, each consisting of a tuple $\langle f(x) \mid x \in X \rangle$ and the associated value of the Choquet integral, the neural network is capable of solving the inverse problem, the problem of determining the fuzzy measure with respect to which the integration takes place.

The second method is also based on the Choquet integral, but it involves Sugeno λ-measures, which capture only a subset of belief and plausibility measures. It solves the inverse problem by a suitable genetic algorithm.

6.2 Uncertainty Measure

Shortly before work on this project began, a new *measure of uncertainty and information in DST* was developed. This measure was shown, contrary to all its competitors, to satisfy all mathematical properties required for such a measure: *subadditivity, additivity, proper range*, and *appropriate boundary conditions* [Harmanec and Klir 1994]. For each finite frame of discernment X and each belief function defined on $\mathcal{P}(X)$, the measure of uncertainty is a function AU defined by

$$AU(Bel) = \max\left\{ -\sum_{x \in X} p_x \log_2 p_x \right\},$$

where the maximum is taken over all probability distributions $\langle p_x \mid x \in X \rangle$ that satisfy for all $A \in \mathcal{P}(X)$ the constraints

$$Bel(A) \leq \sum_{x \in A} p_x .$$

Since function AU is formulated as the solution to a nonlinear optimization problem, its utility appeared initially rather bleak. Fortunately, a relatively simple and fully general algorithm for computing the function was discovered and its correctness proven when working on this project [17]. The existence of this algorithm made the measure practical and opened a way for developing generalized, DST-based information theory [19,29,31, 32].

The algorithm for computing the uncertainty measure AU is formulated as follows:
Input: a frame of discernment X, a belief function Bel on X.
Output: AU(Bel), $\{p_x\}_{x \in X}$ such that $AU(Bel) = -\sum_{x \in X} p_x \log_2 p_x, p_i \geq 0, \sum_{x \in X} p_x = 1,$
and $Bel(A) \leq \sum_{x \in A} p_x$ for all $\emptyset \neq A \subseteq X.$

Step 1. Find a non-empty set $A \subseteq X$, such as $p_x = \dfrac{Bel(A)}{|A|}$ is maximal. If there are more

such sets A, take one with maximal cardinality.

Step 2. For $x \in A$, put $p_x = \dfrac{Bel(A)}{|A|}$.

Step 3. For each $B \subseteq X - A$, put $Bel(B) = Bel(B \cup A) - Bel(A)$.

Step 4. Put $X = X - A$.

If $X \neq \varnothing$ and $Bel(X) > 0$, then go to Step 1.

Step 6. If $Bel(X) = 0$ and $X \neq \varnothing$, then put $p_x = 0$ for all $x \in X$.

Step 7. Calculate $AU(Bel) = -\sum_{x \in X} p_x \log_2 p_x$.

Stop.

In addition to this general algorithm, a special and very efficient algorithm was also developed that is applicable only to possibilistic bodies of evidence [3, 17].

6.3 Uncertainty Principles

Although the utility of the well-justified measure of uncertainty in DST is as broad as the utility of any relevant measuring instrument, its role is particularly pronounced in three fundamental principles for dealing with uncertainty and the associated uncertainty-based information. These principles are: *principle of minimum uncertainty*, *principle of maximum uncertainty*, and *principle of uncertainty invariance* [5]. The first two principles, when formulated within DST, are generalizations of the well-known principles of minimum and maximum entropy of classical, probability-based information theory [Christensen 1985, 1986].

The third uncertainty principle, the principle of uncertainty invariance, is of relatively recent origin [Klir 1990]. It was introduced to facilitate meaningful transformations between the various uncertainty theories. According to this principle, the amount of uncertainty (and the associated uncertainty-based information) should be preserved in each transformation of uncertainty from one mathematical framework to another.

The uncertainty-invariance principle was first applied to *probability–possibility transformations*. During the initial investigation, unfortunately, no well justified measure of possibilistic uncertainty was available. Hence, the results were recently revised, as part of this project, in terms of the well-justified measure AU (Sec. 6.2) and, in addition, transformations from DST to possibility theory (possibilistic approximations of belief measures) were analyzed [3,18,30,32].

The principle of uncertainty invariance is applicable, for example, for obtaining meaningful probabilistic or possibilistic approximations of given bodies of evidence in DST, or for obtaining meaningful approximations of fuzzy sets by crisp sets. The motivation to apply the principle to formalizing meaningful transformations between probabilistic and possibilistic formalization of the same situation has arisen not only from our desire to comprehend the relationship between the two theories of uncertainty, but

also from practical problems. Examples of these problems are: constructing a membership grade function of a fuzzy set from statistical data, combining probabilistic and possibilistic information in expert systems, constructing a probability measure from a given possibility measure in the context of decision making or systems modeling, or converting given probabilities to possibilities to reduce computational complexity.

7. OTHER ISSUES RELEVANT TO IMPRECISE PROBABILITIES

7.1 Fuzzy Measure Theory

In addition to the investigations regarding the various strategies for constructing fuzzy measures, in particular belief measures and Sugeno measures (Sec. 6.1), some important theoretical results were obtained in fuzzy measure theory. These results, which are relevant to imprecise probabilities to various degrees, involve, for example, some theoretical issues regarding the natural extension of lower probabilities [26,37] or the roles of various nonlinear integrals (Sugeno integrals, Choquet integrals, pan-integrals, etc.) in dealing with imprecise probabilities [9,14,15,27,28]. Since results in this category are overly technical and involve many distinct issues, we do not attempt to summarize them here to avoid a presentation that would inevitably be prohibitively long. It seems more prudent in this case to refer to the above listed papers for full coverage of the issues involved.

7.2 Constrained Fuzzy Arithmetic

In the process of developing Bayesian inference with fuzzy probabilities (Sec. 5.2), it became evident that standard fuzzy arithmetic [Kaufman and Gupta 1985] is not adequate. For any finite set of elementary events, probabilities of the events are required to add to 1. When the numerical probabilities are approximated by fuzzy probabilities, the sum of these fuzzy probabilities is not 1 by the rules of standard fuzzy arithmetic. In order to deal with fuzzy probabilities in a meaningful way, the requirement that they always add to exactly 1, regardless how imprecise they are, must be added to fuzzy arithmetic as a requisite constraint.

After recognizing the requisite constraint imposed on arithmetic operations with fuzzy probabilities by axioms of probability theory, one realizes that there are also other requisite constraints that must be considered when using fuzzy arithmetic. Perhaps the most common of them is the requisite equality constraint: given any arithmetic expression with fuzzy intervals, those fuzzy intervals represented by the same symbol (i.e., expressing approximately the value of the same linguistic variable) must be treated as numerically equal.

Any constraint associated with fuzzy intervals on which we operate contains some information. When constraints are ignored, as in standard fuzzy arithmetic, the obtained results are invariably information deficient, which means that they are overly imprecise. This feature of standard fuzzy arithmetic to produce results with inflated imprecision made some practitioners abandon fuzzy arithmetic as impractical.

To guarantee that imprecision be not inflated when applying arithmetic operations to fuzzy intervals, all constraints regarding the fuzzy intervals involved must be taken into account. This may include indigenous constraints (such as the equality constraints) as well as various extraneous constraints (such as the constraint imposed by probability theory when dealing with fuzzy probabilities).

Incorporating constraints into fuzzy arithmetic results, unfortunately, in greater computational complexity. This is due to the fact that arithmetic expressions subjected to constraints must be evaluated globally and not by appropriately sequenced local arithmetic operations. Moreover, the global evaluation can often be accomplished only by solving specific optimization problems [43].

Constrained fuzzy arithmetic has not been sufficiently developed as yet, even though basic issues pertaining to it have already been identified [34, 35, 43]. Some challenging mathematical and computational problems will have to be solved to make it fully operational and useful within the broader area of soft computing [Bonissone 1997, Wang and Tan 1997]. Fortunately, a lot can be learned by examining the many relevant results that have already been obtained in the area of interval computation [Neumaier 1990, Kearfott and Kreinovich 1996].

7.3 Mathematics of Finite Resolution

Imprecise probabilities are used as rough estimates whenever we do not have the capability to obtain precise probabilities. There are various ways in which imprecise probabilities can be expressed: intervals of real numbers, fuzzy intervals, convex sets of probability distributions, rough sets, etc. This raises an interesting question: Is there some underlying theory, by which these seemingly dissimilar characterizations of imprecision of probabilities (or some other quantities of interest) could be treated in a unified way? This issue was investigated in the context of this project. It was found that such a theory, for which the name *theory of finite resolution* was coined, is possible and, when adequately developed, would enhance our understanding of the phenomena of imprecision and uncertainty.

In this study [20], the concept of a finite resolution was defined as a process that returns increasingly better approximations to an object defined by a function or a set of functions from some appropriate function space. The process is formalized as a sequence of operators parameterized by a resolution parameter, which quantifies our ability to discern details. A few basic ideas pertaining to the theory were developed and some problems to

which it could be applied were identified [21]. However, this work is only loosely connected with the objective of this project and, therefore, it was not pursued further within the project. It is expected that further research on the theory will be resumed under a more fitting project in the future.

7.4 Critical Examination of Cox's Proof

The claim that the only correct way to deal with uncertainty is to use the rules of probability theory has been justified by referring to a sequence of intuitive and formal arguments presented by Cox [1946]. These arguments are usually referred to in the literature as the Cox's proof, even though the arguments are not presented as a proof of mathematically stated theorem. The aim of the Cox's proof is to show that the product rule of classical Bayesian inference can be derived mathematically as the only possible operation by properly formalizing our intuitions about rational reasoning. Cox argues that these intuitions can be expressed in terms of specific functional equations whose solution is the product rule. However, the proof is based on some hidden assumptions, which are not explicitly stated by Cox. Although these assumptions have already been used as a basis for criticizing the proof from various points of view, none of these criticisms is fully comprehensive. As a result, the ongoing debate regarding this very fundamental issue, as summarized in [41], became considerably clouded. To clarify the meaning of the Cox's theorem, its comprehensive examination was undertaken in the context of this project [8,41]. Two important results were obtained by this examination: (i) the Cox's proof does not support the alleged claims that the rules of probability are the only coherent rules to deal with uncertainty; and (ii) there exist meaningful Bayesian-like inference rules for dealing with probabilities of fuzzy events, which are different from the classical Bayesian inference.

8. CONCLUSIONS

The purpose of this section is twofold. (i) To identify the principal results emerging from the work on this project and to assess their significance; and (ii) To identify new problem areas, which were opened by some of the results, and assess their importance for future research projects.

The following are principal results, conveniently clustered according to similarities in their meaning:

1. Several efficient methods were developed for Bayesian inference with interval-valued or fuzzy probabilities, which are well-grounded in theory [10,24,25,33,38]. The significance of these methods is that they produce posterior probabilities with exact lower and upper bounds or with exact membership functions defining the respective fuzzy intervals. No comparable method, among those described in the literature, is capable of achieving this; all these methods can produce only various estimates of the

lower and upper bounds or membership functions of posterior probabilities. The methods developed in this project for Bayesian inference with imprecise probabilities are applicable to any problem area where classical Bayesian inference is applicable in principle, but its actual use is compromised since precise prior probabilities or likelihoods cannot be obtained. This utility of the methods is illustrated in [33] by two problem areas: the problem area of military unit identification and the problem area of statistical decision making. The first problem area was chosen as a response to the following statement in *Rome Laboratory Final Technical Report* No. RL-TR-95-99 : "(The Bayes' theorem) does not account for issues, such as errors in observation vectors or uncertainties in prior assumptions utilized by the probability models. So the problem is how to identify enemy military units in battlefield through the observed imprecise information and imprecise prior knowledge of enemy military doctrine." The second problem area, statistical decision making, was chosen because of its broad applicability.

2. Computationally efficient methods were developed for determining imprecise probabilities, either from human judgment or from given data. For eliciting imprecise probabilities from domain experts, a counterpart of Saaty's method of hierarchical analysis [Saaty 1980] was developed, which is based on differences rather than ratios in comparing relevant alternatives and criteria [24, 33]. The significance of this method is that its computational complexity is considerably lower in comparison with Saaty's original method and, hence, its practical utility is significantly higher. For constructing imprecise probabilities (and also some more general fuzzy measures) from data, several methods based on special neural networks or genetic algorithms were developed [11,39,40,42]. The significance of these methods is their computational efficiency, as well as their capability to obtain good approximate solutions when no exact solutions exist and , hence, when analytic methods are not applicable.

3. An efficient algorithm was developed for computing a previously discovered well-justified measure of uncertainty and uncertainty-based information in Dempster-Shafer theory (DST) [17]. Such an algorithm was essential for making the measure practicable and, hence, to make the prospective development of DST-based information theory feasible. The algorithm and the principle of uncertainty invariance [5] were also utilized to formulate operational procedures for sound transformations between possibility theory and probability theory, as well as procedures for transformations from DST to possibility theory or probability theory [3,18,30,32]. In addition, a special function was established for the first time on sound axiomatic grounds for measuring nonspecificity of convex fuzzy subsets of the n-dimensional Euclidean space [6]. This function is applicable, for example, for measuring information obtained by any imprecise measurements regarding the location of an object in space.

4. A careful mathematical analysis was made of hidden assumptions in the well known proof by Cox [1946], whose aim was to justify the classical Bayesian inference as the

sole agent for reasoning under uncertainty [2,41]. This analysis has important implications since it shows that the proof is invalid and does not block alternative rules of inference, distinct from the classical Bayesian rules, for Bayesian-like inference based on fuzzy events. Although some meaningful Bayesian-like inference rules have already been derived and are presented in [41], further research is needed to develop this new area.

5. As a byproduct of research on imprecise probabilities within the broad framework of fuzzy measure theory, numerous special mathematical results were obtained by which fuzzy measure theory has significantly been enhanced [14,15,16,27,28].

6. Several computer programs were developed to support the methodology developed in this project for dealing with imprecise probabilities. Some preliminary steps were also taken to develop a shell program, equipped with a user-friendly interface, in which these programs and other relevant programs will be embedded.

The work in this project produced not only specific results, as presented in publications listed in Sec. 10, but it also opened new avenues for future research. Among them, the following seems to be the most significant new areas of research that emerged from the work on this project:

1. The algorithm for efficient computation of uncertainty in DST [17] makes it feasible not only to operationalize the principles of maximum and minimum uncertainty in DST [5], but also to develop the calculus of *generalized DST-based information theory*.

2. *Constrained fuzzy arithmetic* [22,34,35], which emerged from research on imprecise probabilities in this project, is essential to virtually all applications of linguistic variables that involve arithmetic operations. Some results were already obtained in this new area [34,35], but much more remains to be done. In particular, relevant optimization problems needed to facilitate fuzzy arithmetic under recognized typed of constraints (as emerging from various applications) must be formulated and methodologically developed for this purpose.

3. While basic ideas of the broad *theory of finite resolution* were formulated in the context of this project [20] and shown useful [21], extensive effort will be needed to fully develop it and explore its promising applications to some basic problems of measurement and computation.

4. Alternative rules for *Bayesian-like inference based on fuzzy events*, which emerged from a critical analysis of the Cox's proof [2,41], need to be further studied and their applications examined. In particular, counterparts of the probability calculus need to be developed for the alternative rules of inference.

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APPENDIX: Computer Software

The purpose of this Appendix is to describe basic characteristics of computer programs implementing the various algorithms developed in this project. For each algorithm, these characteristics are: the problem solved by the algorithm, requested input, produced output, relevant restrictions, basic steps of the algorithm, and references to publications where the algorithm is described in detail and supported by a relevant proof of its validity. The use of each program is also illustrated by a simple example.

A. Bayesian Inference with Imprecise Probabilities

In this section, four programs are described, each implementing a special algorithm for *Bayesian* inference with imprecise probabilities. These programs, referred to as Bayesian 1 through *Bayesian* 4, differ from one another in the type of prior probabilities (interval-valued or fuzzy) and in the type of likelihoods (precise, interval-valued, fuzzy), which they accept as inputs, as well as in the type of the calculated posterior probabilities.

A.1 Bayesian 1: Precise Likelihoods and Interval-Valued Prior Probabilities, Interval-valued Posterior Probabilities [33, 38]

Problem
Given interval-valued prior probabilities and precise likelihoods, calculate interval-valued posterior probabilities.

Inputs
- finite universal set: $X = \{x_1, x_2, ..., x_s\}$
- precise likelihoods on X: $a_1, ...,a_s$
- reasonable prior probability intervals on X: $[l_1, u_1], ...,[l_s, u_s]$

Output
- posterior probability intervals on X: $[l_1', u_1'], ...,[l_s', u_s']$

Restrictions
- The sum of the lower prior probabilities $l_1, ...,l_s$ must be less than or equal to one and the sum of the prior upper probabilities $u_1, ...,u_s$ must be greater than or equal to 1.

Basic Steps
Step 1: Based on the given reasonable TPI, compute the corresponding feasible TPI $\langle F_1, ..., F_s \rangle$.
Step 2:. Using the precise likelihoods $a_1, ...,a_s$ and the feasible TPI $\langle F_1, ..., F_s \rangle$ of prior probabilities, calculate the posterior TPI $\langle F_1', ..., F_s' \rangle$.

Example [38]

1. Define a universal set by entering its elements x_1, x_2, x_3, x_4.
2. Specify the following likelihood assignment:

	a_i
x_1	0.25
x_2	0.20
x_3	0.30
x_4	0.10

3. Specify the following prior probability interval assignment:

	l_i	u_i
x_1	0.20	0.25
x_2	0.10	0.50
x_3	0.40	0.45
x_4	0.10	0.15

4. Execute **Bayesian 1**. The resulting output specifying the posterior TPI is:

	l_i'	u_i'
x_1	0.2041	0.2632
x_2	0.1237	0.2500
x_3	0.4 949	0.5625
x_4	0.0404	0.0638

A.2 Bayesian 2: Prior Probabilities, Likelihoods, and Posterior Probabilities are Interval-Valued [33, 38]

Problem
Given interval-valued prior probabilities and likelihoods, calculate interval-valued posterior probabilities.

Inputs
- finite universal set : $X = \{x_1, x_2, ..., x_s\}$.
- likelihood intervals : $[a_1, b_1]$, ...,$[a_s, b_s]$
- reasonable prior probability intervals: $[l_1, u_1]$, ...,$[l_s, u_s]$

Output

- posterior probability intervals: $[l_1', u_1']$, ...,$[l_s', u_s']$

Restrictions

- The sum of the lower probabilities must be less than or equal to one and the sum of the upper probabilities must be greater than or equal to 1.

Basic Steps

Step 1: For the given TPI of reasonable prior probabilities, compute the corresponding feasible TPI $\langle F_1, ...,F_s \rangle$.

Step 2: Using $\langle F_1, ...,F_s \rangle$ and the given likelihood intervals $\langle G_1, ...,G_s \rangle$, calculate the posterior TPI $\langle F_1', ...,F_s' \rangle$.

Example [25]

1. Define a universal set by entering its elements x_1, x_2, x_3, and x_4.
2. Make the following likelihood interval assignment:

	a_i	b_i
x_1	0.20	0.30
x_2	0.10	0.25
x_3	0.30	0.50
x_4	0.10	0.20

3. Make the following prior probability interval assignment:

	l_i	u_i
x_1	0.20	0.25
x_2	0.10	0.50
x_3	0.40	0.45
x_4	0.10	0.15

1. Execute **Bayesian 2**. The resulting output specifying the posterior TPI is:

	l_i'	u_i'
x_1	0.1151	0.3261
x_2	0.0435	0.3061
x_3	0.4324	0.7500
x_4	0.0278	0.1395

A.3 Bayesian 3: Prior Probabilities are Trapezoidal Fuzzy Numbers, Likelihoods are Precise Numbers, and Posterior Probabilities are Trapezoidal Fuzzy numbers [25, 33]

Problem
Given fuzzy prior probabilities and precise likelihoods, calculate fuzzy posterior probabilities.

Inputs
- finite universal set: $X = \{x_1, x_2, ..., x_s\}$.
- precise likelihoods: $a_1, ..., a_s$
- reasonable prior TFP of trapezoidal type: $\langle\langle a_1, b_1, c_1, d_1 \rangle, ..., \langle a_s, b_s, c_s, d_s \rangle\rangle$

Output
- feasible posterior TFP: $\langle\langle a_1', b_1', c_1', d_1' \rangle, ..., \langle a_s', b_s', c_s', d_s' \rangle\rangle$

Restrictions
- for all prior probabilities, $a_i < b_i < c_i < d_i$,
- for all α-cuts of each prior probability, the sum of the lower bounds must be ≤ 1 and the sum of the upper bounds must be ≥ 1.

Basic Steps
Step 1: Calculate the α-cuts of the given fuzzy prior probabilities for all $\alpha \in (0,1]$.

Step 2: Calculate the TPI $\langle {}^\alpha F_1, ..., {}^\alpha F_s \rangle$ of the feasible α-cuts of the prior fuzzy probabilities.

Step 3: Using the given precise likelihoods $\langle a_1, ..., a_s \rangle$ and the feasible α-cuts $\langle {}^\alpha F_1, ..., {}^\alpha F_s \rangle$, calculate the posterior α-cuts $\langle {}^\alpha F_1', ..., {}^\alpha F_s' \rangle$.

Step 4: Calculate trapezoidal approximations of the posterior fuzzy probabilities represented by the α-cuts obtained in Step 3.

Example [25, 33]
1. Define a universal set by entering its elements x_1, x_2, x_3, and x_4.
2. Make the following likelihood assignment:

	a_i
x_1	0.10
x_2	0.20
x_3	0.20
x_4	0.30

3. Make the following probability assignment:

	a_i	b_i	c_i	d_i
x_1	0.00	0.10	0.10	0.20
x_2	0.10	0.20	0.20	0.25
x_3	0.20	0.30	0.30	0.40
x_4	0.30	0.40	0.40	0.60

4. Execute **Bayesian 3**. The resulting output is:

	a_i'	b_i'	c_i'	d_i'
x_1	0.0000	0.0435	0.0435	0.0942
x_2	0.0744	0.1738	0.1738	0.2369
x_3	0.1512	0.2606	0.2606	0.3787
x_4	0.4008	0.5215	0.5215	0.7237

A.4 Bayesian 4: Prior Probabilities, Likelihoods, and Posterior Probabilities are Trapezoidal Fuzzy Numbers [25, 33]

Problem
Given fuzzy prior probabilities and fuzzy likelihoods, calculate fuzzy posterior probabilities.

Inputs
- finite universal set : $X = \{x_1, x_2, ..., x_s\}$
- fuzzy likelihoods of trapezoidal type: $\langle\langle \dot{a}_1, \dot{b}_1, \dot{c}_1, \dot{d}_1 \rangle, ..., \langle \dot{a}_s, \dot{b}_s, \dot{c}_s, \dot{d}_s \rangle\rangle$
- reasonable TFP of trapezoidal type: $\langle\langle a_1, b_1, c_1, d_1 \rangle, ..., \langle a_s, b_s, c_s, d_s \rangle\rangle$

Output
- feasible posterior TFP of trapezoidal type: $\langle\langle a_1', b_1', c_1', d_1' \rangle, ..., \langle a_s', b_s', c_s', d_s' \rangle\rangle$

Restrictions
- for all trapezoidal likelihoods $\dot{a}_i < \dot{b}_i < \dot{c}_i < \dot{d}_i$
- for all trapezoidal prior probabilities, $a_i < b_i < c_i < d_i$.
- for all α-cuts of each prior probability, the sum of the lower bounds must be ≤ 1 and the sum of the upper bounds must be ≥ 1.

Basic Steps

Step 1: Calculate the α-cuts of the given fuzzy prior probabilities for $\alpha \in (0,1]$.

Step 2: Calculate the TPI $\langle {}^\alpha F_1, ..., {}^\alpha F_s \rangle$ of feasible α-cuts of the prior fuzzy probabilities.

Step 3: Calculate the α-cuts $\langle {}^\alpha G_1, ..., {}^\alpha G_s \rangle$ of fuzzy likelihood $G_1, ..., G_s$.

Step 4: Using the α-cuts $\langle {}^\alpha G_1, ..., {}^\alpha G_s \rangle$ and the feasible α-cuts $\langle {}^\alpha F_1, ..., {}^\alpha F_s \rangle$, calculate the α-cuts $\langle {}^\alpha F_1', {}^\alpha F_2', ..., {}^\alpha F_s' \rangle$ of posterior probabilities.

Step 5: Calculate trapezoidal approximations of the posterior fuzzy probabilities represented by the α-cuts obtained in Step 4.

Example [25]

1. Define a universal set by entering elements x_1, x_2, and x_3.
2. Make the following likelihood assignment:

	$\dot{a}_i$	$\dot{b}_i$	$\dot{c}_i$	$\dot{d}_i$
x_1	0.10	0.25	0.25	0.30
x_2	0.20	0.30	0.30	0.50
x_3	0.30	0.40	0.40	0.60

3. Make the following probability assignment:

	a_i	b_i	c_i	d_i
x_1	0.00	0.00	0.20	0.40
x_2	0.10	0.30	0.40	0.50
x_3	0.40	0.60	0.80	0.90

4. Execute **Bayesian 4**. The resulting output specifying the posterior TFP is:

	a_i'	b_i'	c_i'	d_i'
x_1	0.0000	0.0000	0.0704	0.4366
x_2	0.0119	0.2432	0.3334	0.6577
x_3	0.2918	0.6667	0.7568	0.9881

B. Uncertainty in Dempster-Shafer Theory [17]

Problem

Given a body of evidence represented by a belief function Bel of DST, calculate the associated overall uncertainty.

Inputs

- finite universal set (frame of discernment) : X

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- Bel(A) for each A in the power set of X

Output
- AU(Bel)

Restrictions
- Set function Bel must satisfy all properties of belief measures, as defined in DST.

Basic Steps
See Sec. 6.2, pp. 22-23.

Example [17]
1. Define a universal set by entering its elements *a*, *b*, and *c*.
2. Generate the power set.
3. Specify the following belief measure assignment as input:

A	Bel(A)
{a}	0.26
{b}	0.26
{c}	0.26
{d}	0.00
{a,b}	0.59
{a,c}	0.53
{a,d}	0.27
{b,c}	0.53
{b,d}	0.27
{c,d}	0.27
{a,b,c}	0.87
{a,b,d}	0.61
{a,c,d}	0.55
{b,c,d}	0.55
{a,b,c,d}	1.00

4. Execute **Compute AU**. The resulting output is: 1.93598.

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C. Natural Extension of Lower Probabilities [37]

Problem
Given a finite universal set X, a nonnegative function on X, and a set function
$\mu = P(X) \rightarrow [0,1]$ such that $\mu(\varnothing) = 0$ and $\mu(X) = 1$ (a lower probability), calculate the
natural extension, E(f), of f with respect to μ.

Inputs
- finite universal set: $X = \{x_1, x_2, ..., x_s\}$
- nonegative function f on X
- set function μ on the power set of X

Output
- value E of the natural extension of f with respect to μ
- assignment of values λ_j to all sets E_j of the power set [37]

Restrictions
- $f(x) \in [0,1]$ for all $x \in X$
- $\mu(A) \in [0,1]$ for all A in the power set
- $\mu(\varnothing) = 0$ and $\mu(X) = 1$

Basic Steps
See Ref. [37].

Example [37]
1. Define a universal set by entering its elements named x_1, x_2, and x_3.
2. Define the following function:

X	$f(x_j)$
x_1	0.50
x_2	0.10
x_3	0.70

3. Generate the power set of $\{x_1, x_2, x_3\}$.
4. Define the following set function:

$$\mathcal{P}(X)$$

X_3	X_2	X_1	$\mu(E_j)$
0	0	0	0.00
0	0	1	0.20
0	1	0	0.30
0	1	1	0.40
1	0	0	0.10
1	0	1	0.20
1	1	0	0.60
1	1	1	1.00

5. Execute **Natural Extension**. The resulting output is:

- $E = 0.71$

-

X_3	X_2	X_1	λ_j
0	0	0	0.00
0	0	1	0.00
0	1	0	0.30
0	1	1	0.00
1	0	0	0.00
1	0	1	0.20
1	1	0	0.00
1	1	1	0.50

D. Construction of λ-Fuzzy Measures From Given Input-Output Data [11, 39]

Problem
Given a set of input-output pairs, where each input is an n-tuple of real numbers in [0, 1] that represent values of a function f defined on a set X with n elements and each output is a real number in [0, 1] that is viewed either as the Choquet integral or the fuzzy integral of f with respect to an unknown λ-fuzzy measure μ, determine μ that conforms to the given data as closely as possible in terms of the root-mean-square error.

Inputs
- finite universal set: $X = \{x_1, x_2, \ldots, x_s\}$
- input-output data
- Choquet integral or fuzzy integral

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Outputs

- value of parameter λ, by which a λ-fuzzy measure is characterized
- values of measure μ for all singleton sets
- root-mean-square error
- number of iterations employed

Restrictions

- All entries in input-output data must be real numbers in [0, 1].

Basic Steps

A special genetic algorithm is employed; see Ref. [39] for details.

Example

Several examples are described in Ref. [39]. Due to the random search employed in the genetic algorithm, these examples are not exactly reproducible. Hence, we do not present them here.

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